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The Intimate Relation Between the Extended Jaffe Class and Special Spaces of Type S

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Abstract

It is shown that a natural but useful generalisation of Jaffe's mathematical results concerning strict localisability is just an easy consequence of Gelfand's and Shilov's work on spaces of 'type S'. Furthermore, proof is given for the statement that there is no minimal space of test functions with compact support, whereas every countable intersection of such spaces also contains test functions with compact support.

1. *Introduction*

It is well known that fields have to be treated as operator-valued generalised functions, i.e. fields have to be smeared with sufficiently smooth test functions in order to yield proper operators. Although one usually assumes a field to be an operator-valued *tempered* distribution, this is a matter of convenience rather than of conviction. Indeed, there are several hints (see Jaffe (1967) and references given there) indicating that classes of test functions smaller than $\mathscr{S}$ have to be used for physically relevant theories in order to allow the off-mass-shell vacuum expectation values in momentum space to grow faster than any polynomial. This means we have to postulate some sufficiently rapid decrease at infinity for our test functions in momentum space. However, in order to still be able to formulate local commutativity, the question arises whether test functions exist with compact support fulfilling the postulate of rapid decrease in momentum space.

This problem has been treated most successfully by Jaffe (1967), who showed that for an entire function of the form

$$g(t^2) = \sum_{r=0}^{\infty} c_{2r} t^{2r}; \qquad c_0 > 0, \qquad c_{2r} \geq 0 \tag{1.1}$$

the necessary and sufficient condition for the existence of an element $f(x)$ of $\mathcal{D}(R_4)$ fulfilling ($\tilde{f} = \mathcal{F} f$, $\mathcal{F}$: Fourier-transformation)

$$\sup_{p\in R_4} g(A\|p\|^2)(1+\|p\|^2)^n \left|\left(\frac{\partial}{\partial p_0}\right)^{m_0} \cdots \left(\frac{\partial}{\partial p_3}\right)^{m_3} \tilde{f}(p)\right| < \infty \qquad (1.2)$$

for every set of non-negative integers $A, n, m_0, \ldots, m_3$ is given by

$$\int_1^\infty \frac{\log g(t^2)}{t^2}\, dt < \infty \qquad (1.3)$$

Let us denote the class of all $f \in \mathcal{S}(R_4)$ fulfilling (1.2) by $\mathcal{C}_g(R_4)$ and its one-dimensional variant by $\mathcal{C}_g(R_1)$.

Recently, new interest in the Jaffe class of strictly localisable fields, i.e. fields defined over some $\mathcal{C}_g(R_4)$ with g according to (1.1) and (1.3), arose in connection with non-polynomial Lagrangian field theories (Lehmann & Pohlmeyer, 1971; Isham *et al.*, 1971). There is no doubt Jaffe's concept of 'strict localisability' is of the greatest importance. Therefore the test function spaces of Jaffe's class should, at last, be treated in the framework adequate to them: the Gelfand–Shilov theory of spaces of type S. This is, besides some generalisations, the main purpose of the present paper. It sheds some light into the mathematical concept of the Jaffe class and certainly makes work within this class easier.

2. *Formulation of the Mathematical Problem and Results*

A. In order to derive Jaffe's results in an even more general form it is quite sufficient to treat the following simplified problem:

Determine the necessary and sufficient conditions on a non-decreasing positive function g (defined over $[0,+\infty)$) for the existence of an element $f \in \mathcal{D}$ the Fourier transform $\tilde{f} = \mathcal{F} f$ of which fulfils the inequalities:

> $|g(|t|)\tilde{f}^{(q)}(t)| < C_q$ for every $t \in R_1$ and for every non-negative integer q; C_q being a suitable function of q (2.1)

For purely technical reasons we restrict ourselves to such functions g, which have the property:

> For every positive integer n there is a positive number m_n such that the function $t^{-n}g(t)$ is non-increasing over $(0, m_n]$ and non-decreasing over $[m_n, +\infty)$ (2.2)

Note, however, that this restriction is weaker than Jaffe's restriction (1.1), since we are only interested in the non-trivial case $g \notin \mathcal{O}_M$. A function of type (2.2) but not of type (1.1) is, for instance,

$$g(t) = s_a(t) = \exp(|t|^{1/a}) \qquad (2.3)$$

if a is a positive number greater than 1.

We denote the class of all non-decreasing positive functions g (defined over $[0,+\infty)$) with property (2.2) by $\mathscr{G}$ and the class of all functions $f \in \mathscr{S}$ fulfilling (2.1) by $\mathscr{A}_g$ (or, in more detail, $\mathscr{A}_{g(t)}$). Furthermore we use the notation

$$\mathscr{S}_{g,A} = \bigcap_{b\in(0,A)} \mathscr{A}_{g(bt)}; \qquad \mathscr{S}_{g,A} = \mathscr{F}\mathscr{S}_{g,A}; \qquad \mathscr{D}_{g,A} = \mathscr{D} \cap \mathscr{S}_{g,A}$$

A subset $\mathscr{N}$ of $\mathscr{S}$ is called non-trivial iff there is a function $f \in \mathscr{N}$ which does not vanish identically. The 'extended Jaffe class', then, is the class of all countable intersections $\mathscr{I}$ of spaces $\mathscr{S}_{g,A}$ ($g \in \mathscr{G}$, $A > 0$) such that $\mathscr{D} \cap \mathscr{I}$ is non-trivial. Indeed, the extended Jaffe class contains the Jaffe class

$$\{\mathscr{C}_g | \mathscr{D} \cap \mathscr{C}_g \text{ non-trivial}\}$$

because

$$\mathscr{C}_g = \bigcap_{n,m=0}^{\infty} \mathscr{S}_{(1+t^2)^n g(t)^2, m+1}$$

B. Since $\mathscr{D} \cap \mathscr{F}\mathscr{A}_g$ is non-trivial if and only if $\mathscr{D}_{g,A}$ is non-trivial for $A > 0$, the solution of our problem is given by:

Theorem 1: Let g be an element of $\mathscr{G}$ and let A be a positive constant. Then $\mathscr{D}_{g,A}$ is non-trivial if and only if

$$\int_1^{\infty} \frac{\log g(t)}{t^2}\, dt < \infty \tag{2.4}$$

This theorem immediately gives rise to the following question: 'Is there any maximal $g \in \mathscr{G}$ for which (2.4) holds?' The correct answer is 'no':

Lemma 1: Let $g_1, g_2, \ldots$ be a sequence of elements of $\mathscr{G}$ fulfilling (2.4). Then there is an entire $g \in \mathscr{G}$ which fulfils (2.4) and:

$$\lim_{t\to+\infty} g_k(t)/g(t) = 0 \qquad \text{for } k = 1, 2, \ldots$$

While this lemma shows that every countable intersection of non-trivial spaces $\mathscr{D}_{g,A}$ ($g \in \mathscr{G}$, $A > 0$) is non-trivial, the intersection of all non-trivial spaces $\mathscr{D}_{g,A}$ ($g \in \mathscr{G}$, $A > 0$) is trivial:

Theorem 2: Let $f \in \mathscr{D}$ be a function that does not vanish identically. Then there is a function $g \in \mathscr{G}$ fulfilling (2.4) such that $f \notin \mathscr{D}_{g,2}$.

C. The results listed in B allow the following conclusions:

Corollary 1: The extended Jaffe class is the set of all countable intersections of spaces $\mathscr{S}_{g,A}$ with $g \in \mathscr{G}$ fulfilling (2.4) and $A > 0$.

Corollary 2: There is no minimal (in the set theoretical sense) $\mathscr{F}$ within the extended Jaffe class.

3. *Proof of Theorem 1*

A. By $\mathscr{W}$ we denote the class of all entire $g \in \mathscr{G}$ of the form

$$g(t) = \sum_{k=0}^{\infty} c_k t^k; \qquad c_r > 0, \qquad c_r \geqq c_{r+1}, \qquad r = 0, 1, 2, \ldots \tag{3.1}$$

Now let g be some element of $\mathscr{W}$. Under this condition a function $\tilde{f} \in \mathscr{S}$ belongs to $\mathscr{S}_{g,A}$ if and only if:

> $|t^k \tilde{f}^{(q)}(t)| < C_{q,\delta}(1/A + \delta)^k \, 1/c_k$ for every $t \in R_1$, every $\delta > 0$, and for all non-negative integers k, q; $C_{q,\delta}$ being a suitable function of q and δ (3.2)

In other words, $\mathscr{S}_{g,A}$ is a special space of type S (see Gelfand & Shilov (1964)). Therefore we already know from Gelfand & Shilov (1964) how to manage the proof of Theorem 1. All we have to do is to derive the following:

Lemma 2: Let f be an element of $\mathscr{D}$ (and $g \in \mathscr{W}$). If $f \in \mathscr{D}_{g,A}$ then

> $|f^{(k)}(t)| < C_\delta(1/A + \delta)^{k+n} \, 1/c_{k+n}$ for every $t \in R_1$, every $\delta > 0$, and for every integer k; C_δ being a suitable function of δ (3.3)

holds with $n = 2$. Conversely, if (3.3) holds for $n = 0$ then $f \in \mathscr{D}_{g,A}$.

Proof: For every $f \in \mathscr{S}$ we have the inequality

$$\sup_{t \in R_1} |f^{(k)}(t)| \leqq (2\pi)^{-1/2} \int dt \, |t^k \tilde{f}(t)|; \qquad \tilde{f} = \mathscr{F} f$$

In the special case $f \in \mathscr{D}_{g,A}$ this implies

$$\sum_{k=0}^{\infty} (\lambda A)^{k+2} c_{k+2} \sup_{t \in R_1} |f^{(k)}(t)| \leqq \lim_{n \to \infty} (2\pi)^{-1/2} \int dt t^{-2} \sum_{k=2}^{n} c_k |\lambda A t|^k |\tilde{f}(t)| < \infty$$

for $\lambda \in (0, 1)$ and hence the first statement of Lemma 2 is proved.

Conversely, let us suppose that (3.3) is fulfilled for $n = 0$. Since $f \in \mathscr{D}$ and

$$|[t^q f(t)]^{(k)}| = \left| \sum_{\nu=0}^{q} \binom{k}{\nu} [t^q]^{(\nu)} f^{(k-\nu)}(t) \right| \leqq (q+1) k^q \max_{\nu \leq q} |[t^q]^{(\nu)} f^{(k-\nu)}(t)|$$

we even have

> $|[t^q f(t)]^{(k)}| < C'_{q,\delta}(1/A + \delta)^k \, 1/c_k$ for every $t \in R_1$, every $\delta > 0$, and for all non-negative integers k, q; $C'_{q,\delta}$ being a suitable function of q and δ

and hence

$$(2\pi)^{1/2} \sup_{t \in R_1} |t^k \tilde{f}^{(q)}(t)| \leqq \int dt \, |[t^q f(t)]^{(k)}| \leqq D C'_{q,\delta}(1/A + \delta)^k 1/c_k$$

where D denotes the diameter of supp f. This is already (3.2), i.e. the second statement of Lemma 2 is proved, too.

Let us note, by the way, that this proof also shows that a function $f \in \mathscr{D}$ already belongs to $\mathscr{D}_{g,A}$ if its Fourier transform $\tilde{f} = \mathscr{F} f$ fulfils:

$$\max_{t \in R_1} |t^2 g(b|t|)\tilde{f}(t)| < \infty \qquad \text{for } b \in (0, A)$$

Now, according to a theorem of Carleman and Ostrowski (see Gelfand & Shilov (1964) and reference given there) a necessary and sufficient condition for the existence of a function $f \in \mathscr{D}$ fulfilling (3.3) is:

$$\int_1^\infty \frac{\log \Gamma_n(t)}{t^2} dt < \infty \tag{3.4}$$

where Γ_n denotes the 'Ostrowski function', defined by

$$\Gamma_n(t) = \max_k c_{k+n} t^k; \qquad t \geqq 1$$

That (3.4) is indeed equivalent to (2.4) is shown by the following inequalities (valid for $t > 0$):

$$\Gamma_n(t) = t^{-n} \max_k c_{k+n} t^{k+n} \leqq t^{-n} g(t)$$

$$2\Gamma_n(t) = \sum_{k=0}^\infty 2^{-k} \Gamma_n(t) \geqq \sum_{k=0}^\infty c_{k+n} (t/2)^k = (2/t)^n \left[g(t/2) - \sum_{k=0}^{n-1} c_k (t/2)^k \right]$$

Thus Theorem 1 is proved for entire g of the type (3.1).

B. Let g be an arbitrary element of $\mathscr{G}$ and m_n $(n = 1, 2, \ldots)$ the corresponding sequence defined by (2.2). Then the first thing to notice is that this sequence is non-decreasing and unbounded. On the other hand we have

$$g(t) \leqq (t/m_n)^{n+1} g(m_n) \qquad \text{for } m_n \leqq t \leqq m_{n+1}$$

and

$$g(t) \geqq (t/m_n)^n g(m_n) \qquad \text{for } t \in [0, +\infty) \qquad (n = 1, 2, \ldots)$$

Thus, if we define

$$c_0 = g(0); \qquad c_n = g(m_n)/(m_n)^n, \qquad n = 1, 2, \ldots$$

$$\hat{g}(t) = \sum_{\nu=0}^\infty c_\nu t^\nu$$

we get the inequalities:

$$(1 - \lambda)\hat{g}(\lambda t) \leqq g(t) \leqq [g(m_1)/g(0) + t/m_1]\hat{g}(t) + c_0/m_1$$
$$\text{for } t \in [0, +\infty) \qquad \text{and} \qquad \lambda \in (0, 1)$$

Since both functions, on the right and on the left of $g(t)$, are elements of $\mathscr{W}$ (it is for this reason that c_0/m_1 is added on the right) we see that Theorem 1 is also valid for arbitrary $g \in \mathscr{G}$.

C. As a by-product of the proof for Theorem 1, outlined in sections *A* and *B*, we have the following:

Lemma 3: Let g be an element of $\mathscr{G}$ and let A be a positive constant. Then $\mathscr{D}_{g,A}$ is non-trivial if and only if there is a function $f \in \mathscr{D}$ with a Fourier transform $\tilde{f}$ fulfilling (2.1) for q fixed to $q = 0$.

4. *Proof of Lemma* 1

In section 3*B* we have seen that for every $\breve{g} \in \mathscr{G}$, fulfilling (2.4), there is an entire $g \in \mathscr{W}$, fulfilling (2.4) with

$$\breve{g}(t) \leq g(t) \qquad \text{for } t > 0$$

Hence it is sufficient to prove Lemma 1 for $g_k \in \mathscr{W}$. Then

$$g(t) = t \sum_{k=1}^{\infty} a_k g_k(t) + a.; \qquad \text{for } t \in R_1 \tag{4.1}$$

defines an entire $g \in \mathscr{W}$ with the desired property

$$\lim_{t \to +\infty} g_k(t)/g(t) = 0 \qquad \text{for } k = 1, 2, \ldots$$

if the sequence $a_1, a_2, \ldots$ is non-increasing, positive and of sufficiently rapid decrease; for example:

$$0 < a_k \leq [k! g_k(k)]^{-1}; \qquad a_{k+1} \leq a_k; \qquad k = 1, 2, \ldots \tag{4.2}$$

Since

$$\lim_{\xi \to +0} \int_1^{\infty} \frac{\log[\breve{g}_1(t) + \xi \breve{g}_2(t)]}{t^2} dt = \int_1^{\infty} \frac{\log \breve{g}_1(t)}{t^2} dt$$

for every two functions $\breve{g}_1, \breve{g}_2 \in \mathscr{W}$ fulfilling (2.4), we may impose the additional restrictions

$$\int_1^{\infty} \frac{\log \sum_{\nu=1}^{k} a_\nu g_\nu(t)}{t^2} dt < \sum_{r=1}^{k} (1/2)^r; \qquad k = 1, 2, \ldots \tag{4.3}$$

on the sequence $a_1, a_2, \ldots$. Then the restrictions (4.3) and (4.2) guarantee that g, as defined by (4.1), is an element of $\mathscr{W}$ fulfilling (2.4). Thus Lemma 1 is proved.

5. *Proof of Theorem* 2

Let $f \in \mathscr{D}$ be a function that does not vanish identically. Then

$$a_q = \max_{r \leq q} \sup_{t \in R_1} |f^r(t)| \tag{5.1}$$

defines a non-decreasing sequence of positive numbers $a_0, a_1, \ldots$ Gelfand & Shilov (1964) have shown that $\mathscr{D} \cap S^1$ is trivial (see also their definition and properties of S^1). For f this means:

$$\inf_q \frac{B^q q^q}{a_q} = 0 \qquad \text{for every } B > 0$$

Therefore, if we define recursively

$$c_0 = 1/a_0, \qquad c_k = \min\,[1/a_k, c_{k-1}/k] \tag{5.2}$$

we have

$$\max_{r \geq q} c_r a_r = 1, \qquad q = 1, 2, \ldots \tag{5.3}$$

and hence:

$$\lim_{q \to \infty} \lambda^q c_q a_q = 0 \qquad \text{for } \lambda \in (0, 1) \tag{5.4}$$

While (5.2) tells us that

$$g(t) = \sum_{k=0}^{\infty} c_k t^k$$

is an element of $\mathscr{W}$, we conclude from (5.4) that (3.3) is valid for $n = 0$ and $A = 1$, i.e.:

$$f \in \mathscr{D}_{g,1}$$

Thus, by Theorem 1, g fulfils (2.4). Consequently we get an entire $\hat{g} \in \mathscr{W}$ fulfilling (2.4) if we define

$$\hat{g}(t) = \sum_{k=0}^{\infty} \hat{c}_k t^k; \qquad t \in R_1$$

$$\hat{c}_0 = \hat{c}_1 = \hat{c}_2 = c_0; \qquad \hat{c}_{k+2} = c_k; \qquad k = 0, 1, 2, \ldots$$

and, according to Theorem 1, $\mathscr{D}_{\hat{g},2}$ is non-trivial. But (5.3) and (5.1) show that

$|f^{(k)}(t)| < C_\delta (1/2 + \delta)^{k+2}\, 1/\hat{c}_{k+2}$ for every $t \in R_1$, every $\delta > 0$, and for every integer k; C_δ being a suitable function of δ

is not valid. Hence, by application of Lemma 1 (first statement), we see that

$$f \notin \mathscr{D}_{\hat{g},2}$$

Thus Theorem 2 is proved.

6. *Discussion*

A. To give an illustrative example for the application of our results, let us note that (in the Gelfand–Shilov notation (1964)) $S^{\beta,1} = \widetilde{S_{\beta,1}}$ is in the extended Jaffe class for $\beta > 1$ and that the space $\mathscr{C}(R_1)$, as introduced by Jaffe in 1966, is just the intersection of all $S^{\beta,1}$ with $\beta > 1$. Hence Corollary 1 tells us that $\mathscr{C}(R_1)$ is in the extended Jaffe class.

Another application is the following: Suppose

$$\sum_{\nu=0}^{\infty} c_\nu \delta^{(2\nu)}(t)$$

to be a generalised function over some $\mathscr{C}_g$ of Jaffe's class. Then there is a non-trivial $f \in \mathscr{D} \cap \mathscr{C}_g$, for which we may conclude:

$$\begin{aligned}\infty > \sum_{\nu=0}^{\infty} |c_\nu (f * f)^{2(\nu)}(0)| &= \sum_{\nu=0}^{\infty} |c_\nu| \max_{t\in R_1} |(f * f)^{2(\nu)}(t)| \\ &\geq \sum_{\nu=0}^{\infty} |c_\nu| \max_{t\in R_1} \left| \int (f * f)^{2(\nu)}(t') \mathrm{e}^{itt'} \, dt' \right| D^{-1} \\ &= (2\pi)^{1/2} D^{-1} \sum_{\nu=0}^{\infty} |c_\nu| \max_{t\in R_1} t^{2\nu} \tilde{f}^2(t) \\ &\geq (2\pi)^{1/2} D^{-1} \max_{t\in R_1} \left| \tilde{f}^2(t) \sum_{\nu=0}^{\infty} |c_\nu| \, t^2 \right|\end{aligned}$$

where D denotes the finite diameter of $\operatorname{supp} f * f$. Hence, by Lemma 3 and Theorem 1,

$$g(t^2) = \sum_{\nu=0}^{\infty} |c_\nu| \, t^2$$

defines an entire function of type (1.3). This argument provides the substance for Lehmann's and Pohlmeyer's definition of 'minimally singular' super-propagators in non-polynomial Lagrangian field theory (Lehmann & Pohlmeyer, 1971).

B. Although in the present work we have considered the one-dimensional case only, generalisation to the n-dimensional case $(n > 1)$ is quite easy (Jaffe, 1967). Furthermore, since $h \in \mathscr{D}$ and $f \in \mathscr{C}_g$ imply $h * f^2 \in \mathscr{C}_g$, it is a consequence of a distribution theoretical standard argument that every space $\mathscr{C}_g$ of the Jaffe class is dense in $\mathscr{D}$. Thus it is justifiable to say that our results are more general than Jaffe's.

C. As far as the physical applications are concerned, there are several things to be improved in Jaffe, 1966 and 1967. For example, Jaffe's definition of strict localisability is not related to the topological structure of the test function space, whereas one should (without reducing the class) postulate $\mathscr{D} \cap \mathscr{C}_g$ to be dense in $\mathscr{C}_g$. An unpleasant feature of the Jaffe class is, for instance, that the lack of Fourier symmetry implies that there are no Lorentz-invariant multipliers in $\mathscr{C}_g(R_4)$ $(g \notin \mathscr{O}_M)$ other than polynomials. However, all these questions will be the subject of a forthcoming paper (Lücke, in preparation).

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